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2004 J. Phys. A: Math. Gen. 37 9771

(http://iopscience.iop.org/0305-4470/37/41/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 9771–9789 PII: S0305-4470(04)77640-4

Galilean covariant Lagrangian models

E S Santos[1,2], M de Montigny[1,3], F C Khanna[1,4] and A E Santana[5]

[1] Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, T6G 2J1, Canada
[2] Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona, 145 São Paulo, SP, 01405-900, Brazil
[3] Faculté Saint-Jean, University of Alberta, Edmonton, Alberta, T6C 4G9, Canada
[4] TRIUMF, 4004, Wesbrook Mall, Vancouver, British Columbia, V6T 2A3, Canada
[5] Instituto de Física, Universidade de Brasilia, Brasilia, DF, 70910-900, Brazil

E-mail: esantos@phys.ualberta.ca, montigny@phys.ualberta.ca, khanna@phys.ualberta.ca and santana@fis.unb.br

Received 13 March 2004, in final form 2 September 2004
Published 29 September 2004
Online at stacks.iop.org/JPhysA/37/9771
doi:10.1088/0305-4470/37/41/011

Abstract

We construct non-relativistic Lagrangian field models by enforcing Galilean covariance with a (4, 1) Minkowski manifold followed by a projection onto the (3, 1) Newtonian spacetime. We discuss scalar, Fermi and gauge fields, as well as interactions between these fields, preparing the stage for their quantization. We show that the Galilean covariant formalism provides an elegant construction of the Lagrangians which describe the electric and magnetic limits of Galilean electromagnetism. Similarly we obtain non-relativistic limits for the Proca field. Then we study Dirac Lagrangians and retrieve the Lévy-Leblond wave equations when the Fermi field interacts with an Abelian gauge field.

PACS numbers: 03.50.De, 11.10.Ef, 11.10.Kk, 11.30.Cp

1. Introduction

Recent successes in low temperature condensed matter physics have motivated further investigation of Galilean invariance, because such phenomena are typically non-relativistic. Moreover, the algebraic structure underlying Galilean invariance is, somewhat paradoxically, more intricate than its relativistic counterpart. For instance, it admits a non-trivial central extension, so that the physical states are described by projective representations [1]. Therefore, any method which would simplify its formulation, or make it similar to relativistic theories, is likely to be useful. The sharing of concepts between relativistic and non-relativistic field theories is not new: one simply has to think of the very concept of field, the Higgs mechanism of spontaneous symmetry breaking, the Goldstone boson, etc. This is the general line of thought which we pursue in this paper. Specifically, in the same way that Lorentz covariance provides a simple algorithm to construct relativistic field equations, here we use Galilean covariance as a guiding principle to obtain field equations for non-relativistic theories. This is done by embedding the (3, 1) Newtonian spacetime in a Minkowski manifold $\mathcal{G}_{(4,1)}$. That this is a promising procedure is illustrated in section 4 by a remarkably simple and natural construction of the Lagrangians which lead to the so-called electric and magnetic Galilei-invariant limits of electromagnetism [2]. This involves auxiliary fields, and the form of the Lagrangians, far from obvious if constructed by hand, is quite straightforward when we work within $\mathcal{G}_{(4,1)}$. A few articles related to Galilean field theories are given in [3].

In this paper, we perform a systematic investigation of the Galilean covariant formulation of non-relativistic Lagrangian field models. We utilize the method and notation found in [4–6]. Similar formulations can be found in the literature [7–10]. Essentially, this approach exploits the fact that the eleven-dimensional centrally extended Galilei algebra in (3, 1) spacetime is a subalgebra of the fifteen-dimensional Poincaré algebra in (4, 1) spacetime [11]. Hereafter we will not review it in detail, and instead refer to the literature [4–6], as well as the early sections of our own articles [12–14], and references therein. The approach consists in writing down Lagrangians defined on the manifold $\mathcal{G}_{(4,1)}$, and then performing an appropriate reduction to the (3, 1) Newtonian spacetime. Also, a corresponding projection is to be done at the level of the fields. In group theoretical terms, this is explained by embedding representations of the extended Galilei group of (3, 1) spacetime into the Poincaré group in $\mathcal{G}_{(4,1)}$ [11]. The problem of superfluous components is similar to the polarization components of photons. For massless fields, one usually reduces the number of components from 4 to 2 with the Lorentz gauge condition and some additional choice. For massive spin 1 fields, we would be down from 5 to 3 with a gauge condition and some choice. The tensor formalism utilized henceforth has been devised explicitly in section 3 of [6]. In addition to treating Lorentz and Galilean spacetime on the same footing, this approach provides a promising tool for the path integral [15] and canonical quantization of non-relativitic field theories [16].

The manifold $\mathcal{G}_{(4,1)}$ is such that a Galilean boost acts on 5-vectors $X = (\mathbf{X}, X^4, X^5)$ as

$$\mathbf{X}' = \mathbf{X} - \mathbf{V}X^4, \qquad X'^4 = X^4, \qquad X'^5 = X^5 - \mathbf{V}\cdot\mathbf{X} + \tfrac{1}{2}\mathbf{V}^2 X^4 \tag{1}$$

where $\mathbf{V}$ is the relative velocity. Various motivations for the definition of X^5 are found in [6, 8, 12–16]. The scalar product, $(X|Y) = X^\mu Y_\mu \equiv \mathbf{X}\cdot\mathbf{Y} - X_4Y_5 - X_5Y_4$, of two 5-vectors, X and Y, is invariant under the transformations given in equation (1). This suggests that we define Galilean tensor calculus by using the *Galilean metric*:

$$g_{\mu\nu} = \begin{pmatrix} \mathbf{1}_{3\times3} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Henceforth, we shall use the following embedding of the Galilean spacetime into $\mathcal{G}_{(4,1)}$:

$$(\mathbf{x}, t) \hookrightarrow x^\mu = (x^1, \ldots, x^5) \equiv (\mathbf{x}, t, s) \tag{2}$$

except in section 4, where the fourth component will be denoted by ct. We identify the 5-momentum as

$$p_\mu = -\mathrm{i}\partial_\mu = (-\mathrm{i}\nabla, -\mathrm{i}\partial_t, -\mathrm{i}\partial_s) = (\mathbf{p}, -\mathcal{E}, -m) \tag{3}$$

so that $p^4 = -p_5 = m$, the mass, and $p^5 = -p_4 = \mathcal{E}$, the energy. Note also that the relation $\partial_s = -\mathrm{i}m$ implies that, for a complex field $\Phi(x)$ defined on $\mathcal{G}_{(4,1)}$, the corresponding field $\varphi(\mathbf{x}, t)$, projected on (3, 1) spacetime, is defined by the ansatz

$$\Phi(x) \equiv \mathrm{e}^{-\mathrm{i}ms}\varphi(\mathbf{x}, t). \tag{4}$$

The extra coordinate s may be related to the quasi-invariance of the free particle Lagrangian under Galilean transformations, or to the phase of the wavefunction that guarantees the Galilean invariance of the Schrödinger equation [6]. An elegant interpretation was given by Kapuścik, who proposed that the additional coordinate s be required as some control parameter due to the absence of signal with universal velocity in Galilean physics [8]. Following the usual definition of space and time with rods, clocks, mirrors and signals, Kapuścik defines s in terms of the two velocities (from the emitter to the mirror, and from the mirror to the emitter) which appear in the synchronization procedure, thereby claiming to achieve completeness of Galilean physics through operational definitions of space and time.

Since the relativistic (3, 1) Minkowski space is also clearly contained within $\mathcal{G}_{(4,1)}$, this formalism has the desirable feature that it allows one to treat relativistic or non-relativistic theories in an unified approach, depending on how the (3, 1) spacetime is embedded into $\mathcal{G}_{(4,1)}$, as first emphasized in [9, 10] (see also the last section of [6]). The underlying vector space is clearly a (4, 1) Minkowski space with a light-cone-like change of basis [6]. If $(\mathbf{x}, x^4, x^5)$ is a Galilean 5-vector, with geometry described by the Galilean metric, then the 5-vector

$$\left(\mathbf{x}, \tau = \frac{x^4 + x^5}{\sqrt{2}}, \xi = \frac{x^4 - x^5}{\sqrt{2}}\right) \tag{5}$$

is associated with the diagonal metric diag$(1, 1, 1, -1, 1)$ (section 3 of [6]). Thus, the projection of a Galilean model, described by the embedding of equation (2), to a relativistic theory, can be performed by letting

$$(\mathbf{x}, t) \hookrightarrow x^\mu = (x^1, \ldots, x^5) \equiv \left(\mathbf{x}, \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right). \tag{6}$$

Then, from equation (5), we have $\tau = t$ and $\xi = 0$. This process can be reversed, if one wishes to exploit known results concerning relativistic theories, and obtain the corresponding results for Galilean theories, by using equation (5). This approach is very promising because many relativistic results could then lead to a better understanding of non-relativistic systems. This will be illustrated with a simple example in section 3.

The paper is organized as follows. In section 2, we give a general discussion of the Lagrangian formalism in $\mathcal{G}_{(4,1)}$ and its symmetries. The Galilean Klein–Gordon complex field which corresponds, after dimensional reduction, to the Schrödinger field, is the subject of section 3. In section 4, we discuss the electromagnetic Lagrangians which describe the magnetic and electric limits of Le Bellac and Lévy-Leblond. The massive spin 1 Proca field is also considered. Finally, the Fermi field is discussed in section 5. Lowercase Latin indices a, b, c, etc denote three-dimensional Euclidean coordinates $1, 2, 3$, whereas Greek indices μ, ν, etc run from 1 to 5. Uppercase Latin indices A, B, etc represent field components, transformation parameters and gauge indices.

2. Lagrangians in Galilean manifold

Consider a general action functional defined on $\mathcal{G}_{(4,1)}$:

$$I[\Phi] = \int_R \mathrm{d}^5x\, \mathcal{L}[\Phi(x), \partial_\mu \Phi(x)] \tag{7}$$

depending on n fields Φ_A, with $A = 1, \ldots, n$. The integral is over an arbitrary five-dimensional volume R within $\mathcal{G}_{(4,1)}$. Henceforth, we shall omit the index A. Each field $\Phi(x)$ is a function of x^μ, with $\mu = 1, \ldots, 5$. The extra coordinate s is defined over the real numbers, so that any integral over s can be interpreted as

$$\int \mathrm{d}x^5 \to \lim_{l\to\infty} \frac{1}{2l} \int_{-l}^{l} \mathrm{d}s. \tag{8}$$

Therefore, an integral over R will be reduced to the usual integral over (3, 1) spacetime if the integrand is independent of s.

Hereafter, we shall investigate the symmetries of the action, following well-known methods found in the literature [17]. Variations of coordinates, $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$, and field coordinates,

$$\Phi(x) \to \Phi'(x) = \Phi(x) + \delta_0\Phi(x) \tag{9}$$

(where $\delta_0\Phi(x)$ denotes the functional change of $\Phi(x)$), together with the principle of stationary action, $\delta I[\Phi] = 0$, and the assumption that the variations of the fields and coordinates vanish at the boundary, lead to the Euler–Lagrange field equations:

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} = 0 \tag{10}$$

for each field Φ. As mentioned above, the centrally extended Galilei group is a subgroup of the group of inhomogeneous Lorentz transformations over $\mathcal{G}_{(4,1)}$, that is, $\delta x^\mu = \epsilon^{\mu\nu}x_\nu + a^\mu$. Therefore, we will consider these particular transformations. For later use, let us define the *local* variation of the field as

$$\delta\Phi(x) = \Phi'(x') - \Phi(x) = \delta_0\Phi(x) + \delta x^\mu\partial_\mu\Phi(x) \tag{11}$$

where $\delta_0\Phi(x)$ is defined in equation (9). The epithet 'local' is used because, unlike equation (9), x and x' in equation (11) represent the same geometrical point.

Now let us turn to the transformations which leave the action invariant, and identify the corresponding conserved quantities. We write the infinitesimal form of the group of transformations that leave the action I invariant as

$$\delta x^\mu = (M_x)^\mu_A\delta\omega^A, \qquad \delta\Phi = (M_\Phi)_A\delta\omega^A \tag{12}$$

where M_x and M_Φ are arrays which define the transformations, and A labels the transformation parameters $\delta\omega$. If we assume that the Euler–Lagrange equations (10) are satisfied, then the action's response to these variations may be cast into the form

$$\delta I = 0 = \int_{\partial R} \mathrm{d}s_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}(M_\Phi)_A - T^\mu{}_\nu\,(M_x)^\nu_A\right]\delta\omega^A \tag{13}$$

where

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta^\mu{}_\nu\mathcal{L} \tag{14}$$

is the energy–momentum tensor. Therein a summation over the fields Φ is understood. The same applies if Φ is complex; in this case the sum is over its real and complex parts. From Schwinger's fundamental postulate [17], equation (13) may be written as $\delta I = Q_A[2] - Q_A[1]$, where 1, 2 denote spacelike hypersurfaces in $\mathcal{G}_{(4,1)}$, with Q_A being the generators of the field transformations. Then we have

$$\Phi'(x) = \mathrm{e}^{\mathrm{i}\delta\omega^A Q_A}\Phi(x)\,\mathrm{e}^{-\mathrm{i}\delta\omega^A Q_A} \approx \Phi(x) + \mathrm{i}\delta\omega^A[Q_A, \Phi(x)] + \mathcal{O}((\delta\omega)^2) \tag{15}$$

so that, from equation (9), we find

$$\delta_0\Phi(x) = \mathrm{i}\delta\omega^A[Q_A, \Phi(x)]. \tag{16}$$

From Gauss' theorem, $\int_{\sigma=\partial R}\omega = \int_R \mathrm{d}\omega$, we find that equation (13) leads to a continuity equation,

$$\partial_\mu\mathcal{J}^\mu{}_A = 0 \tag{17}$$

where the conserved current is given by the integrand in equation (13):

$$\mathcal{J}^{\mu}{}_{A} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}(M_\Phi)_A - T^{\mu}{}_{\nu}(M_x)^{\nu}_{A}. \tag{18}$$

Equations (13) and (17) give rise to a time-independent conserved charge defined by $Q_A = \int_{\sigma=\partial R} \mathcal{J}^{\mu}{}_{A}\, \mathrm{d}s_\mu$ where $\sigma = \partial R$ denotes a hypersurface. If this hypersurface is chosen as $x^4 =$ constant, then the conserved charge reduces to $Q_A = \int_R \mathcal{J}^4{}_A\, \mathrm{d}^3x\, \mathrm{d}x^5$ and we retrieve Noether's theorem, $\frac{\mathrm{d}Q_A}{\mathrm{d}t} = 0$. This also shows that the parameter $x^5 = s$ can be integrated out if $\mathcal{J}^4{}_\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_4\Phi)}(M_\Phi)_\mu - T^4{}_\nu (M_x)^\nu{}_\mu$ does not depend on s.

For example, consider invariance under coordinate translations, for which equation (12) reads

$$\delta x^\mu = \delta\omega^\mu \text{ (constant)}, \qquad \delta\Phi = 0. \tag{19}$$

Since $A = 1, \ldots, 5$, we may replace it with the coordinate Greek indices. Thus, we have $(M_x)^\mu{}_\nu = \delta^\mu{}_\nu$ and $(M_\Phi)^\mu{}_\nu = 0$, so that, from equation (18), we find $\mathcal{J}^\mu{}_\nu = -T^\mu{}_\nu$. Then equation (17) implies that

$$\partial_\mu T^\mu{}_\nu = 0. \tag{20}$$

If the factor of $\partial_5\Phi(x)$ within the Lagrangian density takes the form $\mathrm{e}^{+\mathrm{i}ms} f(\mathbf{x}, t)$ after equations (2) and (4) have been substituted, then equation (20) takes the form $\partial_a T_{a\mu} + \partial_t T_{5\mu} = 0$, that is, the last term, $\partial_s T^5{}_\mu$, vanishes. This is the case for the models studied in this paper. This leads to the following interpretations of the components of the energy–momentum tensor: $-T_{ab}$ is the density of momentum flux of the field, T_{5a} is the field momentum density, which may be seen as the density of mass flux in the direction a; T_{a4} is its density of energy flux, $-T_{45}$ the energy density, $-T_{55}$ is the density of mass. The component T_{44}, however, has no interpretation in terms of physical quantities in (3, 1) Newtonian spacetime.

Let us denote the conserved charge by

$$P_\mu = Q_\mu = -\int_{\sigma=\partial R} \mathrm{d}^3x\, \mathrm{d}x^5\, T^4{}_\mu = \int_V \mathrm{d}x^3\, T_{5\mu}, \qquad \mu = 1, \ldots, 5 \tag{21}$$

where we have used equation (8), and with V a three-dimensional volume. From equation (14), we have $T^4{}_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_t \Phi)}\partial_\mu \Phi - g^4{}_\mu \mathcal{L}$. Recalling that $g^\mu{}_\nu = \delta^\mu{}_\nu$, we see that for $\mu = 4$, we have $T^4{}_4 = \frac{\partial \mathcal{L}}{\partial(\partial_t\Phi)}\partial_t\Phi - \mathcal{L}$. Therefore, $T^4{}_4 = -T_{54}$ may be identified with the energy density of the system, as mentioned previously. Equation (9) becomes $\delta_0\Phi(x) = \Phi'(x) - \Phi(x) = \Phi(x - \delta\omega) - \Phi(x) \approx -\delta\omega^\mu \partial_\mu\Phi(x)$, since $\Phi'(x') = \Phi(x)$, so that $\Phi'(x) = \Phi(x - \delta\omega)$. This result, together with equation (16), leads to the commutation relation

$$[\Phi(x), P_\mu] = -\mathrm{i}\partial_\mu \Phi(x). \tag{22}$$

Now let us turn to the angular momentum and Galilean boosts, since the field models considered in this paper have their action invariant under Lorentz rotations in $\mathcal{G}_{(4,1)}$. They are defined by restricting equation (12) to

$$\delta x^\mu = \epsilon^{\mu\nu} x_\nu, \qquad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu}, \qquad |\epsilon^{\mu\nu}| \ll 1 \tag{23}$$

where $\epsilon^{\mu\nu}$ plays the role of $\delta\omega$ in equation (12), as well as

$$\delta\Phi = \tfrac{1}{2} K_{\mu\nu}\epsilon^{\mu\nu}\Phi(x) \tag{24}$$

where, for each $\mu\nu$ pair, $K_{\mu\nu}$ is a representation matrix of the (4, 1) Poincaré group acting on the field Φ, i.e. $\delta\Phi_A(x) = \frac{1}{2}\epsilon^{\mu\nu} K^{AB}_{\mu\nu}\Phi_B(x)$, where A, B are representation indices. In equation (12), this amounts to replacing A with a double index $\mu\nu$, so that by comparing equations (12) and (23), we find $(M_x)^\mu{}_{\alpha\beta} = \frac{1}{2}\left(\delta^\mu{}_\alpha x_\beta - \delta^\mu{}_\beta x_\alpha\right)$. If we insert equation (24) into (12), then we find $(M_\Phi)_{\alpha\beta} = \frac{1}{2}K_{\alpha\beta}\Phi(x)$. If we substitute this equation and the previous one into (18), then we obtain the conserved current corresponding to Lorentz transformations

$$\mathcal{J}^\mu{}_{\alpha\beta} = \frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}K_{\alpha\beta}\Phi(x) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}(x_\beta\partial_\alpha - x_\alpha\partial_\beta)\Phi(x) + \left(\delta^\mu{}_\alpha x_\beta - \delta^\mu{}_\beta x_\alpha\right)\mathcal{L}\right]. \tag{25}$$

And if we define the angular momentum tensor as $\mathcal{M}^{\mu\alpha\beta} = 2\mathcal{J}^{\mu\alpha\beta}$, and, in a way similar to equation (21), we relate it to the components of the angular momentum by

$$M^{\mu\nu} \equiv \int \mathrm{d}^3x\, \mathrm{d}x^5\, \mathcal{M}^{4\mu\nu} \tag{26}$$

then we can show that these are the generators of the Lorentz transformations, by proceeding as for the translations. These calculations lead to

$$[\Phi(x), M^{\mu\nu}] = \mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\Phi(x) + \mathrm{i}K^{\mu\nu}\Phi(x). \tag{27}$$

The commutation relations involving P^μ and $M^{\mu\nu}$ are computed by using the fact that, given $[\Phi(x), X]$ for $X = \Theta_A$ and Θ_B, then $[\Theta_A, \Theta_B]$ may be obtained from Jacobi identity: $[\Phi(x), [\Theta_A, \Theta_B]] = [[\Phi(x), \Theta_A], \Theta_B] - [[\Phi(x), \Theta_B], \Theta_A]$. This leads to the commutation relations of the 15-dimensional Poincaré algebra of (4, 1) spacetime:

$$\begin{aligned}
&[M^{\mu\nu}, M^{\alpha\beta}] = \mathrm{i}(g^{\mu\alpha}M^{\nu\beta} + g^{\nu\beta}M^{\mu\alpha} - g^{\nu\alpha}M^{\mu\beta} - g^{\mu\beta}M^{\nu\alpha})\\
&[P^\mu, M^{\alpha\beta}] = -\mathrm{i}(g^{\mu\alpha}P^\beta - g^{\mu\beta}P^\alpha)\\
&[P^\mu, P^\nu] = 0.
\end{aligned} \tag{28}$$

The 11-dimensional extended Galilean algebra of (3, 1) spacetime is the subalgebra which consists of

$$\begin{aligned}
&M_{ab} \to \epsilon_{abc}J_c && \text{rotations}\\
&M_{4a} \to K_a && \text{Galilei boosts}\\
&P_a \to P_a && \text{space translations}\\
&P_4 \to -H && \text{time translations}\\
&P_5 \to -m\mathbf{1}
\end{aligned} \tag{29}$$

where $a, b, c = 1, 2, 3$. In other words, the Galilei subalgebra is obtained simply by selecting the coordinate indices different from 5. This amounts to removing the four generators M_{a5} and M_{45}. The definitions of the two generators P_4 and P_5 are compatible with equation (3). The non-zero commutation relations of the Galilei Lie algebra are

$$\begin{aligned}
&[J_a, J_b] = \mathrm{i}\epsilon_{abc}J_c, && [J_a, K_b] = \mathrm{i}\epsilon_{abc}K_c\\
&[J_a, P_b] = \mathrm{i}\epsilon_{abc}P_c, && [K_a, H] = \mathrm{i}P_a\\
&[K_a, P_b] = \mathrm{i}\delta_{ab}m\mathbf{1}.
\end{aligned} \tag{30}$$

As mentioned above, the mass m, which appears as the central charge, is a remnant of P_5. It would be interesting to investigate the (2, 1) Newtonian spacetime. In this case, the central extension of the Galilei Lie algebra admits two central charges, rather than one, as pointed by Lévy-Leblond in (1971) [1]. That the interpretation of this second central charge is not so clear is illustrated in [18].

3. Klein–Gordon Lagrangian and Schrödinger field

The simplest example of a relativistically invariant wave equation can be obtained from the Galilean Klein–Gordon Lagrangian

$$\mathcal{L}_{\mathrm{GKG}} = -\frac{1}{2m}(\partial^\mu\Phi^*\partial_\mu\Phi - k^2|\Phi|^2) - V(|\Phi|). \tag{31}$$

By ‘Galilean’, we mean that the field is defined on $\mathcal{G}_{(4,1)}$, and this model will describe non-relativistic physics, once we have defined the embedding of equation (2). The Euler–Lagrange field equations with respect to Φ^* give the scalar equation

$$\frac{1}{2m}(\partial^\mu\partial_\mu + k^2)\Phi = \frac{\delta V}{\delta\Phi^*}. \tag{32}$$

With the embedding of equations (2) and (4), and absorbing k into the energy operator, this becomes the Schrödinger equation

$$\mathrm{i}\partial_t\varphi = -\frac{1}{2m}\nabla^2\varphi + \frac{\delta V}{\delta\varphi^*}. \tag{33}$$

The more familiar Schrödinger equation, $\mathrm{i}\partial_t\varphi = -\frac{1}{2m}\nabla^2\varphi + \mathcal{V}(r)\varphi$, may be obtained by restricting the potential to $V(|\varphi|) = -\mathcal{V}(r)|\varphi|^2$. Note that if, rather than first finding the Euler–Lagrange equations, we begin by substituting equations (2) and (4) into $\mathcal{L}_{\mathrm{GKG}}$, then it becomes

$$\mathcal{L} = -\frac{1}{2m}|\nabla\varphi|^2 - \frac{\mathrm{i}}{2}((\partial_t\varphi^*)\varphi - \varphi^*\partial_t\varphi) - V(|\varphi|). \tag{34}$$

The variation with respect to φ^* leads, once again, to the Schrödinger equation (33). In the next sections, we will see that for other models, the resulting equations of motion are not the same, depending on the order in which we define the embedding or compute the Euler–Lagrange equations. For the quartic self-interaction, $V(|\Phi|) = \frac{1}{2}\lambda\Phi^4$, equation (33) becomes

$$\mathrm{i}\partial_t\varphi = -\frac{1}{2m}\nabla^2\varphi + \lambda|\varphi|^2\varphi. \tag{35}$$

This is referred to as the non-linear Schrödinger equation or, in condensed matter physics, as the Gross–Pitaevskii equation.

The energy–momentum tensor, equation (14), takes the symmetric form

$$T_{\mu\nu} = -\frac{1}{2m}[\partial_\mu\Phi^*\partial_\nu\Phi + \partial_\mu\Phi\partial_\nu\Phi^* - g_{\mu\nu}(\partial^\alpha\Phi^*\partial_\alpha\Phi - k^2|\Phi|^2)] + g_{\mu\nu}V(|\Phi|)) \tag{36}$$

with the Galilean metric $g_{\mu\nu}$. From the embedding (2) and the field representation (4), the components of this tensor read

$$\begin{aligned}
T_{ab} = T_{ba} &= -\frac{1}{2m}(\partial_a\varphi^*\partial_b\varphi + \partial_a\varphi\partial_b\varphi^* - \delta_{ab}|\nabla\varphi|^2)\\
&\quad + \frac{\mathrm{i}}{2}\delta_{ab}((\partial_t\varphi^*)\varphi - \varphi^*\partial_t\varphi) + \delta_{ab}\left(-\frac{k^2}{2m}|\varphi|^2 + V(|\varphi|)\right),\\
T_{4a} = T_{a4} &= -\frac{1}{2m}(\partial_t\varphi^*\partial_a\varphi + \partial_t\varphi\partial_a\varphi^*),\\
T_{5a} = T_{a5} &= -\frac{\mathrm{i}}{2}(\varphi^*\partial_a\varphi - \varphi\partial_a\varphi^*),\\
T_{45} = T_{54} &= -\frac{1}{2m}|\nabla\varphi|^2 - \frac{1}{2m}k^2|\varphi|^2 - V(|\varphi|),\\
T_{44} &= -\frac{1}{m}\partial_t\varphi^*\partial_t\varphi, \qquad T_{55} = -m|\varphi|^2.
\end{aligned} \tag{37}$$

The physical interpretation of each component is discussed below equation (20). As usual, we perform a Fourier decomposition and use a Dirac delta function. From equations (21) and (3), we find the following expression for the field 5-momentum

$$P_\mu = \int \mathrm{d}^3p\, p_\mu|\tilde{\varphi}(p)|^2. \tag{38}$$

As mentioned in the introduction, the manifold $\mathcal{G}_{(4,1)}$ also contains the (3, 1)-dimensional Minkowski space so that a different embedding leads to relativistic equations. Indeed, if we use the embedding (6) within equation (4), then $\mathcal{L}_{\mathrm{GKG}}$, with $V = 0$ and $k = m$, reduces to the manifestly relativistic expression (except for a multiplicative factor of $2m$)

$$\mathcal{L} = \nabla\varphi^* \cdot \nabla\varphi - \partial_t\varphi^*\partial_t\varphi - m^2|\varphi|^2. \tag{39}$$

This corroborates the fact, already mentioned in the introduction, that our formalism unifies relativistic and non-relativistic theories. This also implies that equation (5) may be exploited to obtain non-relativistic results from known relativistic ones. Specifically, we illustrate how a solution of the relativistic Klein–Gordon equation in (4, 1) dimensions may lead to a solution of the non-relativistic Schrödinger equation in three dimensions. Now let us illustrate that with plane wave solutions of equation (32), with V constant. With the coordinates $x = (\mathbf{x}, \tau, \xi)$ of equation (5), this differential equation reads

$$\nabla^2\Phi(x) - \partial_{\tau\tau}\Phi(x) + \partial_{\xi\xi}\Phi(x) + k^2\Phi(x) = 0. \tag{40}$$

If we write the 5-momentum of the $\tilde{x}_\mu = (\mathbf{x}, \tau, \xi)$ basis as $\tilde{p}_\mu = (\mathbf{p}, \omega, \kappa)$, then the plane wave solution reads

$$\Phi(x) = C\,\mathrm{e}^{\mathrm{i}\tilde{p}_\mu\tilde{x}^\mu} = C\exp\mathrm{i}(\mathbf{p}\cdot\mathbf{x} - \omega\tau + \kappa\xi) \tag{41}$$

where C is a constant, with the dispersion relation

$$\tilde{p}^\mu\tilde{p}_\mu = \mathbf{p}^2 - \omega^2 + \kappa^2 = k^2. \tag{42}$$

With the change of coordinates, equation (5), the solution (41) becomes

$$\Phi(x) = C\exp\mathrm{i}\left(\frac{\omega-\kappa}{\sqrt{2}}\right)s\,\exp\mathrm{i}\left(\mathbf{p}\cdot\mathbf{x} + \frac{\omega+\kappa}{\sqrt{2}}t\right). \tag{43}$$

Next, we identify the first factor with the definition in equation (4), so that $\frac{\omega-\kappa}{\sqrt{2}} = -m$ and $\frac{\omega+\kappa}{\sqrt{2}} = -\mathcal{E}$. This allows us to see that the dispersion relation (42) is

$$\mathbf{p}^2 - 2m\mathcal{E} = k^2 \tag{44}$$

Thus we may write equation (43) as

$$\Phi(x) = C\,\mathrm{e}^{\mathrm{i}p_\mu x^\mu} \tag{45}$$

where we have used again equations (2) and (3) with the Galilean metric. The factor $\varphi(\mathbf{x}, t)$ in equation (4) is easily identified as $\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{x}-\mathcal{E}t)}$, that is, the solution of the free particle Schrödinger equation (33), as claimed. Whether this procedure allows us to obtain or classify more intricate solutions is an open question, and deserves further investigation.

4. Non-relativistic electromagnetism

In this section, we turn to the simplest gauge theory: electromagnetism. A good illustration of the fact that there is more to Galilean invariance than just taking the ratio v/c very small is illustrated in [19]. Hereafter, we retrieve Galilei-invariant electric and magnetic limits [2] of electromagnetism by using the tensorial form of Maxwell equations, and determine the Lagrangian densities from which the following field equations are derived:

$$\begin{aligned}
&\nabla\cdot\mathbf{B} = 0\\
&\nabla\cdot\mathbf{E}_m = \frac{1}{\epsilon_0}\rho_m\\
&\nabla\times\mathbf{B} = \mu_0\mathbf{J}\\
&\nabla\times\mathbf{E}_m = -\partial_t\mathbf{B}
\end{aligned} \tag{46}$$

for the 'magnetic' limit, and

$$
\begin{aligned}
&\nabla \cdot \mathbf{B} = 0 \\
&\nabla \cdot \mathbf{E}_e = \frac{1}{\epsilon_0}\rho_e \\
&\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0\partial_t \mathbf{E}_e \\
&\nabla \times \mathbf{E}_e = \mathbf{0}
\end{aligned}
\tag{47}
$$

for the 'electric' limit. Note that the displacement current term is missing in the third line of equation (46), and that the Faraday induction term does not appear in the last line of equation (47). The purpose of Le Bellac and Lévy-Leblond was to write down the laws of electromagnetism by enforcing Galilean relativity rather than Einstein's relativity. Therefore, the equations above could have been formulated during the pre-relativity era.

Their starting point is that the Lorentz transformation of a four-vector $(u^0, \mathbf{u})$:

$$
u'^0 = \gamma \left(u^0 - \frac{1}{c}\mathbf{V}\cdot\mathbf{u}\right) \qquad \mathbf{u}' = \mathbf{u} - \gamma\frac{\mathbf{V}}{c}u^0 + \frac{\mathbf{V}}{\mathbf{V}^2}(\gamma - 1)\mathbf{V}\cdot\mathbf{u} \tag{48}
$$

where $\gamma \equiv \frac{1}{\sqrt{1-\mathbf{V}^2/c^2}}$, with relative velocity $\mathbf{V}$, admits two well-defined Galilean limits. The speed of light in the vacuum is denoted by c. One limit is related to largely timelike vectors, with $u'^0 = u^0$ and $\mathbf{u}' = \mathbf{u} - \frac{1}{c}\mathbf{V}u^0$, and it corresponds to the electric limit. The second limit is for largely spacelike vectors, which have $u'^0 = u^0 - \frac{1}{c}\mathbf{V}\cdot\mathbf{u}$ and $\mathbf{u}' = \mathbf{u}$, and is associated with the magnetic limit. The magnetic limit corresponds to systems were the magnetic field, multiplied by the velocity of light, is much greater than the electric field. The opposite situation, where the electric field is large, corresponds to the electric limit. In addition to the field equations, Le Bellac and Lévy-Leblond have determined various field transformations, but they have not discussed which Lagrangians provide the two Galilean limits [2]. To our knowledge, this question has not yet been addressed in the literature. Therefore, the main outcome of this section is an elegant answer to this question. We will see that the two Lagrangians have the same form, and both involve different auxiliary fields, which are set equal to zero once the equations of motion have been obtained. Also, this example illustrates the fact that the concept of embedding, and the definition of the fields in terms of the extra coordinate, s, are not trivial. In particular, we will see that the resulting equations of motion depend on whether we calculate the Euler–Lagrange equations before or after the field projection (from $\mathcal{G}_{(4,1)}$ to the (3, 1) Newtonian spacetime) is performed.

In this section, we modify the embedding of equation (2) in such a way that all the components have units of length:

$$
(\mathbf{x}, t) \hookrightarrow x^\mu = (x^1, \ldots, x^5) \equiv (\mathbf{x}, ct, s) \tag{49}
$$

where c has the dimension of a velocity. Despite this suggestive notation, c is not the speed of light. Equation (49) implies that we must replace equation (3) with

$$
p_\mu = -\mathrm{i}\partial_\mu = \left(-\mathrm{i}\nabla, -\frac{\mathrm{i}}{c}\partial_t, -\mathrm{i}\partial_s\right) = \left(\mathbf{p}, -\frac{\mathcal{E}}{c}, -mc\right) \tag{50}
$$

so that $p^4 = -p_5 = mc$ and $p^5 = -p_4 = \mathcal{E}/c$. Thus, we obtain $\partial_s = -\mathrm{i}mc$.

The five-dimensional Galilean Lagrangian of electromagnetism, that is, the Maxwell Lagrangian interacting with an external 5-current J_μ is given by

$$
\mathcal{L}_{\mathrm{GEM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{\epsilon_0 c}J_\mu A^\mu \tag{51}
$$

where

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{52}
$$

When we perform a variation with respect to the gauge fields A^{μ}, we obtain the field equations:

$$\partial_{\mu} F_{\alpha\beta} + \partial_{\alpha} F_{\beta\mu} + \partial_{\beta} F_{\mu\alpha} = 0 \tag{53}$$

and

$$\partial_{\mu} F^{\mu\nu} = -\frac{1}{\epsilon_0 c} J^{\nu}. \tag{54}$$

For later convenience, let us introduce the parameter μ_0:

$$\mu_0 \epsilon_0 = \frac{1}{c^2}. \tag{55}$$

In [13], we have shown that these equations lead to the electric and magnetic limits, as identified by Le Bellac and Lévy-Leblond [2]. Moreover we have noticed that the transformation laws of A_{μ}, J_{μ} and the electromagnetic field can be retrieved very naturally with our five-dimensional algorithm. However, one cannot determine the Lagrangians which provide the two Galilean limits, equations (46) and (47), by simply defining the fields as in [13]. Indeed, if we substitute equations (39) and (40) of [13] into equation (51), then we find the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\mathbf{B}_e^2 + \mu_0 \epsilon_0 \partial_t \mathbf{A}_e \cdot \nabla \phi_e + \tfrac{1}{2}\mu_0^2 \epsilon_0^2 (\partial_t \phi_e)^2 - \mu_0 \mathbf{J}_e \cdot \mathbf{A} \tag{56}$$

which clearly does not lead to equation (47). For the magnetic limit, the situation is worse: the electric field does not even appear within the Lagrangian. Substituting equations (49) and (50) of [13] into the Lagrangian (51) leads to

$$\mathcal{L} = -\tfrac{1}{2}\mathbf{B}_m^2 - \mu_0 \mathbf{J}_m \cdot \mathbf{A}. \tag{57}$$

It seems that in order to construct two such Lagrangians, with one leading to the electric limit, and the other leading to the magnetic limit, one needs to introduce auxiliary fields. The latter are used throughout the computation of the Euler–Lagrange equations, and then they may be eliminated. Hereafter, we utilize the formalism based on $\mathcal{G}_{(4,1)}$ to find that the ensuing field equations may be obtained from a single Lagrangian in (4, 1) dimensions which reduces to two different Lagrangians in (3, 1) spacetime. It turns out that one set of auxiliary fields leads to the electric limit, whereas a complementary set of auxiliary fields provides the magnetic limit. Whereas the form of this Lagrangian, as well as the physical and auxiliary fields, are suggested very naturally by using the manifold $\mathcal{G}_{(4,1)}$, it would be far from obvious without this formalism. This is done by defining the 5-potential as

$$A_{\mu}(x) = (\mathbf{A}(\mathbf{x}, t), -\phi_m(\mathbf{x}, t), -\phi_e(\mathbf{x}, t)). \tag{58}$$

That these fields do not depend on s can be traced back, using equation (3), to the fact that they describe a massless field, so that $\partial_s = m = 0$. The 5-current J_{μ} is defined similarily:

$$J_{\mu} = (\mathbf{J}(\mathbf{x}, t), -c\rho_m(\mathbf{x}, t), -c\rho_e(\mathbf{x}, t)) \tag{59}$$

where each component is independent of s.

Let us denote the components of the field strength tensor by

$$F_{\mu\nu} = \begin{pmatrix} 0 & cB_3 & -cB_2 & E_{m1} & E_{e1} \\ -cB_3 & 0 & cB_1 & E_{m2} & E_{e2} \\ cB_2 & -cB_1 & 0 & E_{m3} & E_{e3} \\ -E_{m1} & -E_{m2} & -E_{m3} & 0 & a \\ -E_{e1} & -E_{e2} & -E_{e3} & -a & 0 \end{pmatrix} \tag{60}$$

so that, from equations (52) and (58), we find

$$\begin{aligned} a &= -\frac{1}{c}\partial_t\phi_e \\ c\mathbf{B} &= \nabla\times\mathbf{A} \\ \mathbf{E}_m &= -\nabla\phi_m - \frac{1}{c}\partial_t\mathbf{A} \\ \mathbf{E}_e &= -\nabla\phi_e. \end{aligned} \tag{61}$$

Then, by substituting this into equation (51), the Galilean version of Lagrangian $\mathcal{L}_{\mathrm{EM}}$ reads

$$\mathcal{L}_{\mathrm{GEM}} = -\frac{1}{2}c^2\mathbf{B}^2 + \mathbf{E}_m\cdot\mathbf{E}_e + \frac{1}{2c^2}(\partial_t\phi_e)^2 + \frac{1}{\epsilon_0 c}\mathbf{J}\cdot\mathbf{A} - \frac{1}{\epsilon_0}\rho_m\phi_e - \frac{1}{\epsilon_0}\rho_e\phi_m. \tag{62}$$

This is the central result of this section.

Once again, let us repeat that there are *not* two kinds of physical electric fields, $\mathbf{E}_e$ and $\mathbf{E}_m$. Only one is taken to be the physical field, while the other is an auxiliary field, in the respective (electric or magnetic) limit. If we compute the Euler–Lagrange equations with respect to the fields ϕ_e, ϕ_m and $\mathbf{A}$, we find

$$\nabla\cdot\left(-\nabla\phi_m - \frac{1}{c}\partial_t\mathbf{A}\right) = \frac{1}{\epsilon_0}\rho_m + \frac{1}{c^2}\partial_{tt}\phi_e \tag{63}$$

$$\nabla\cdot(-\nabla\phi_e) = \frac{1}{\epsilon_0}\rho_e \tag{64}$$

and

$$c\nabla\times\mathbf{B} = \frac{1}{\epsilon_0 c}\mathbf{J} + \frac{1}{c}\partial_t(-\nabla\phi_e) \tag{65}$$

respectively.

In order to retrieve Le Bellac and Lévy-Leblond's *magnetic limit* [2], we define the auxiliary quantities ϕ_e and ρ_e as

$$\phi_e = 0, \qquad \rho_e = 0 \tag{66}$$

with $\mathbf{E}_m$ given by equation (61), so that equations (63) and (65) reduce to the Gauss' law,

$$\nabla\cdot\mathbf{E}_m = \frac{1}{\epsilon_0}\rho_m \tag{67}$$

as well as

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \tag{68}$$

respectively. Equation (64) vanishes identically.

The *electric limit* is obtained by defining

$$\phi_m = 0, \qquad \rho_m = 0 \tag{69}$$

with $\mathbf{E}_e$ given by equation (61). Equations (64) and (65) lead to Gauss' law,

$$\nabla\cdot\mathbf{E}_e = \frac{1}{\epsilon_0}\rho_e \tag{70}$$

and

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\partial_t\mathbf{E}_e \tag{71}$$

respectively. From equation (63), we obtain the Lorentz gauge condition

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\partial_t\phi_e = 0. \tag{72}$$

Now let us turn to the homogeneous equations (53), that is,

$$\begin{aligned}
&\nabla\cdot\mathbf{B}=0\\
&\nabla\times\mathbf{E}_m+c\partial_4\mathbf{B}=0\\
&\nabla\times\mathbf{E}_e+c\partial_5\mathbf{B}=0\\
&\nabla a-\partial_4\mathbf{E}_e+\partial_5\mathbf{E}_m=0.
\end{aligned}\tag{73}$$

The first of these clearly leads to

$$\nabla\cdot\mathbf{B}=0\tag{74}$$

in both limits. The second leads, in the magnetic limit defined by equations (61) and (66), to

$$\nabla\times\mathbf{E}_m=-\partial_t\mathbf{B}\tag{75}$$

and vanishes identically in the electric limit, defined by equation (69). The third equation gives

$$\nabla\times\mathbf{E}_e=0\tag{76}$$

in the electric limit, equation (69), and vanishes identically in the magnetic limit, equation (66). The fourth equation leads to an identically vanishing result. To summarize, the magnetic limit of the Maxwell equations (46) is retrieved from equations (67), (68), (74) and (75). The electric limit equations (47) are obtained by combining equations (70), (71), (74) and (76).

Let us mention that the same equations of motion are a result of equation (54), cast into the form

$$\partial_\alpha\partial^\alpha A^\mu=-\frac{1}{\epsilon_0 c}J^\mu\qquad\longrightarrow\qquad\nabla^2A^\mu=-\frac{1}{\epsilon_0 c}J^\mu\tag{77}$$

where we have used the condition $m=0$, together with the five-dimensional Lorentz gauge condition, $\partial_\mu A^\mu=0$, from equation (72) . In the magnetic limit, we find, by substituting equation (66) into this equation, that $\nabla\cdot\mathbf{A}=0$. Instead, if we use equation (66) then we obtain, for the electric limit, $\nabla\cdot\mathbf{A}+\frac{1}{c}\partial_t\phi_e=0$.

Using equation (14) for the Lagrangian of equation (51), we find

$$T_{\mu\nu}=-F_{\mu\rho}\partial_\nu A^\rho-g_{\mu\nu}\mathcal{L}_{\mathrm{EM}}.\tag{78}$$

In order to compare with the relativistic situation [20], let us consider the canonical symmetric energy–momentum tensor, or Belinfante tensor, $\Theta^{\mu\nu}$. It is defined by

$$\Theta^{\mu\nu}\equiv T^{\mu\nu}+\partial_\alpha K^{\alpha\mu\nu}\tag{79}$$

where $K^{\alpha\mu\nu}=-K^{\mu\alpha\nu}$, i.e. it is antisymmetric with respect to the first two indices. Then we have $\partial_\mu\Theta^{\mu\nu}=\partial_\mu T^{\mu\nu}+\partial_\mu\partial_\alpha K^{\alpha\mu\nu}=\partial_\mu T^{\mu\nu}=0$. Using $K^{\alpha\mu\nu}=F^{\alpha\mu}A^\nu$ in equation (79), the symmetric energy–momentum tensor reads [20]

$$\Theta_{\mu\nu}=g^{\alpha\beta}F_{\mu\alpha}F_{\beta\nu}+\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}.\tag{80}$$

Its components read

$$\begin{aligned}
&\Theta_{ab}=c^2B_aB_b+E_{ma}E_{eb}+E_{mb}E_{ea}-\frac{1}{2}\delta_{ab}\left[c^2\mathbf{B}^2+2\mathbf{E}_e\cdot\mathbf{E}_m-\frac{1}{c^2}(\partial_t\phi_e)^2\right]\\
&\Theta_{a4}=c(\mathbf{E}_m\times\mathbf{B})_a-\frac{1}{c}E_{ma}\partial_t\phi_e\\
&\Theta_{a5}=c(\mathbf{E}_e\times\mathbf{B})_a+\frac{1}{c}E_{ea}\partial_t\phi_e\\
&\Theta_{45}=-\frac{1}{2}c^2\mathbf{B}^2-\frac{1}{2c^2}(\partial_t\phi_e)^2\\
&\Theta_{44}=-\mathbf{E}_m^2,\qquad\Theta_{55}=-\mathbf{E}_e^2.
\end{aligned}\tag{81}$$

The remaining components are obtained from the symmetry of $\Theta_{\mu\nu}$.

In the magnetic limit, from equation (66), we have that $\phi_e = 0$, or $\mathbf{E}_e = \mathbf{0}$, and the above equations become

$$\begin{array}{lll} \Theta_{ab} = c^2 B_a B_b - \frac{1}{2}\delta_{ab} c^2 \mathbf{B}^2, & \Theta_{a5} = 0, & \Theta_{a4} = c(\mathbf{E}_m \times \mathbf{B})_a \\ \Theta_{55} = 0, & \Theta_{45} = -\frac{1}{2} c^2 \mathbf{B}^2, & \Theta_{44} = \mathbf{E}_m^2. \end{array} \tag{82}$$

To obtain the electric limit, we utilize equation (69), with $\phi_m = 0$, or $\mathbf{E}_m = -\partial_t \mathbf{A}$, and the components of Θ read

$$\begin{aligned} \Theta_{ab} &= c^2 B_a B_b - \frac{1}{c}(E_{eb}\partial_t A_a + E_{ea}\partial_t A_b) - \frac{1}{2}\delta_{ab}\left[c^2\mathbf{B}^2 - \frac{2}{c}\mathbf{E}_e \cdot \partial_t \mathbf{A} - \frac{1}{c^2}(\partial_t \phi_e)^2\right] \\ \Theta_{a4} &= -(\partial_t \mathbf{A} \times \mathbf{B})_a + \frac{1}{c}(\partial_t \phi_e)\partial_t A_a \\ \Theta_{a5} &= c(\mathbf{E}_e \times \mathbf{B})_a + \frac{1}{c}(\partial_t \phi_e) E_{ea} \\ \Theta_{45} &= -\frac{1}{2}c^2 \mathbf{B}^2 - \frac{1}{2c^2}(\partial_t \phi_e)^2 \\ \Theta_{44} &= -\frac{1}{c^2}(\partial_t \mathbf{A})^2, \qquad \Theta_{55} = -\mathbf{E}_e^2. \end{aligned} \tag{83}$$

The corresponding relativistic result can be obtained using $\tilde{x}^\mu = U^\mu_\nu x^\nu$ where the matrix U performs the transformation (5). Afterwards, we use the embbeding (6). Such a transformation, when applied to the tensor $F_{\mu\nu}$, leads to $\tilde{F} = UFU^{-1}$ and the associated metric is given by $\tilde{g} = UgU^{-1} = \mathrm{diag}(1, 1, 1, -1, 1)$. Using this approach we can recover the relativistic tensor $\tilde{\Theta}_{\mu\nu}$ with additional components $\Theta_{5\mu} = 0$ [20].

Now we discuss the Galilean covariant Proca field, which is a non-relativistic massive vector field. For convenience, we use $c = 1$ henceforth. As for the electromagnetic massless field, we obtain two limits: the magnetic limit was obtained by Lévy-Leblond in [21]. The electric limit is not found in the literature. We start with the covariant Lagrangian:

$$\mathcal{L}_{\mathrm{GProca}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{k^2}{2}A_\mu A^\mu, \tag{84}$$

where the vector A_μ should obey the condition $\partial_\mu A^\mu = 0$, with $k \neq 0$. From this Lagrangian and the definition of F, we get the equation of motion

$$\partial_\mu F^{\mu\nu} + k^2 A^\nu = 0 \quad \rightarrow \quad (\partial_\mu \partial^\mu + k^2)A_\nu = 0 \tag{85}$$

that is, the Schrödinger equation. Using the other way, we can obtain the equations

$$\begin{aligned} &\mathbf{E}_m = -\nabla\phi_m - \partial_t \mathbf{A} \\ &\mathbf{E}_e = -\nabla\phi_e + \mathrm{i}m\mathbf{A} \\ &\mathbf{B} = \nabla \times \mathbf{A} \\ &\nabla \times \mathbf{B} = \partial_t \mathbf{E}_e - \mathrm{i}m\mathbf{E}_m + k^2\mathbf{A} \\ &\nabla \cdot \mathbf{E}_m = \mathrm{i}m\partial_t\phi_m + \partial_{tt}\phi_e + k^2\phi_m \\ &\nabla \cdot \mathbf{E}_e = \mathrm{i}m\partial_t\phi_e - m^2\phi_m + k^2\phi_e \end{aligned} \tag{86}$$

and the general transformation that leaves the Lagrangian invariant is obtained by applying the transformation (1) to the operator ∂_μ and the field A_μ. This gives

$$\begin{aligned} &\mathbf{E}'_m = \mathbf{E}_m + \mathbf{v} \times \mathbf{B} + \mathbf{v}(\partial_t\phi_e + \mathrm{i}m\phi_m) - \mathbf{v}(\mathbf{v} \cdot \mathbf{E}_e) + \frac{\mathbf{v}^2}{2}\mathbf{E}_e \\ &\mathbf{E}'_e = \mathbf{E}_e \\ &\mathbf{B}' = \mathbf{B} - \mathbf{v} \times \mathbf{E}_e. \end{aligned} \tag{87}$$

As mentioned previously, there are not two kinds of physical electric fields. If $A_5 = 0$, then we obtain what we call the magnetic limit, due to the similarity with the massless case. The equations are a particular case of equations (86):

$$\begin{aligned} &\mathbf{E}_m = -\nabla\phi_m - \partial_t \mathbf{A} \\ &\mathbf{B} = \nabla \times \mathbf{A} \\ &\nabla \times \mathbf{B} = \mathrm{i}m\partial_t \mathbf{A} - \mathrm{i}m\mathbf{E}_m + k^2\mathbf{A} \\ &\nabla \cdot \mathbf{E}_m = \mathrm{i}m\partial_t\phi_m + k^2\phi_m \end{aligned} \tag{88}$$

where the equations for the field $\mathbf{E}_e$ have been subtracted because they are redundant identities, and the condition $\partial_\mu A^\mu = 0$ is given by $\nabla \cdot \mathbf{A} - \mathrm{i}m\phi_m = 0$. The transformations for the fields are also obtained similarly from equations (87). This result agrees with [21]. If $A_4 = 0$, then we obtain the electric limit, with equations also obtained from (86):

$$\begin{aligned} &\mathbf{E}_e = -\nabla\phi_e + \mathrm{i}m\mathbf{A} \\ &\mathbf{B} = \nabla \times \mathbf{A} \\ &\nabla \times \mathbf{B} = \partial_t \mathbf{E}_e + \mathrm{i}m\partial_t \mathbf{A} + k^2\mathbf{A} \\ &\nabla \cdot \mathbf{E}_e = \mathrm{i}m\partial_t\phi_e + k^2\phi_e. \end{aligned} \tag{89}$$

Now the equations for the field $\mathbf{E}_m$ are redundant identities and $\partial_\mu A^\mu = \nabla \cdot \mathbf{A} + \partial_t\phi_e = 0$. We do not discuss these two limits any further since their physical interpretations are similar to the massless case. The expression for the mass–energy–momentum tensor is similar to equation (80), with the Lagrangian replaced by equation (84).

5. Interacting Fermi field

The Galilean version of the Dirac equation has been investigated by using the present formalism in [13]. Therein we have retrieved the Lévy-Leblond equations [21], as well as the Pauli equation, spin–orbit interaction and a Darwin-like term. Moreover, a generalized model involving the interaction of a non-Abelian gauge field with the Dirac field has been presented. Hereafter we complete the discussion by examining the related Lagrangian densities.

First, let us consider the Galilean Dirac Lagrangian for the free Fermi field

$$\mathcal{L}_{\mathrm{GDirac}} = \overline{\Psi}(\mathrm{i}\gamma^\mu \overleftrightarrow{\partial}_\mu - k)\Psi \tag{90}$$

where $A\overleftrightarrow{\partial}B \equiv \frac{1}{2}[A\partial B - (\partial A)B]$. We use the following gamma matrices [6]:

$$\gamma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \tag{91}$$

where each entry is a two-by-two matrix and the σ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{92}$$

These gamma matrices satisfy the usual relation: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where $g^{\mu\nu}$ is the Galilean metric. Following [6], we define the adjoint spinor by $\overline{\Psi} = \Psi^\dagger\zeta$, with

$$\zeta = \frac{-\mathrm{i}}{\sqrt{2}}(\gamma^4 + \gamma^5) = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{93}$$

Let us now utilize the embedding within the Lagrangian (90). From the definitions above, with spinor $\Psi \equiv \binom{\psi_1}{\psi_2}$ and equation (4), we find that equation (90) becomes

$$\mathcal{L}_{\rm GDirac} = \frac{1}{2}\big[\big(\nabla\psi_2^\dagger\big)\cdot\sigma\psi_1 - \psi_2^\dagger\sigma\cdot\nabla\psi_1 - \psi_1^\dagger\sigma\cdot\nabla\psi_2 + \big(\nabla\psi_1^\dagger\big)\cdot\sigma\psi_2\big] - \frac{\sqrt{2}}{2}\big(\psi_1^\dagger\partial_t\psi_1 - \big(\partial_t\psi_1^\dagger\big)\psi_1\big) + \mathrm{i}m\sqrt{2}\psi_2^\dagger\psi_2 - \mathrm{i}k\big(\psi_2^\dagger\psi_1 - \psi_1^\dagger\psi_2\big). \tag{94}$$

Variations of this Lagrangian with respect to ψ_1 and ψ_2 lead to

$$\mathrm{i}\sqrt{2}\partial_t\psi_1^\dagger + \mathrm{i}\nabla\psi_2^\dagger\cdot\sigma + k\psi_2^\dagger = 0 \tag{95}$$

and

$$\big(\nabla\psi_1^\dagger\big)\cdot\sigma + \mathrm{i}k\psi_1^\dagger + \mathrm{i}\sqrt{2}m\psi_2^\dagger = 0 \tag{96}$$

respectively. With respect to their conjugates $\psi_1^\dagger$ and $\psi_2^\dagger$, we obtain

$$\mathrm{i}\sqrt{2}\partial_t\psi_1 + (\mathrm{i}\sigma\cdot\nabla + k)\psi_2 = 0 \tag{97}$$

and

$$(\mathrm{i}\sigma\cdot\nabla - k)\psi_1 + \sqrt{2}m\psi_2 = 0 \tag{98}$$

respectively. When we substitute ψ_2 from equation (97) into equation (98), we obtain

$$\mathrm{i}\partial_t\psi_1 = -\frac{1}{2m}(\nabla^2 + k^2)\psi_1. \tag{99}$$

If we absorb the constant k into the energy operator, we clearly obtain equation (33) with a constant potential.

Now let us return to the Lagrangian (90) and find the Euler–Lagrange equations before performing any embedding. The variation of $\mathcal{L}_{\rm GDirac}$ with respect to $\overline{\Psi}$ gives

$$(\mathrm{i}\gamma^\mu\partial_\mu - k)\Psi = 0. \tag{100}$$

The Euler–Lagrange equation with respect to its adjoint gives

$$\overline{\Psi}(\mathrm{i}\gamma^\mu\overleftarrow{\partial}_\mu + k) = 0 \tag{101}$$

where $A\overleftarrow{\partial} \equiv \partial A$. Now, using the embedding (4) into equation (100) leads to equations (97) and (98), whereas with equation (101) we retrieve equations (95) and (96). As usual, by multiplying equation (100) on the left with $\mathrm{i}\gamma^\mu\partial_\mu + k$, we find equation (32), with a constant potential.

The energy–momentum tensor calculated for $\mathcal{L}_{\rm GDirac}$ with equation (14) reads

$$T_{\mu\nu} = \mathrm{i}\overline{\Psi}\gamma_\mu\overleftrightarrow{\partial}_\nu\Psi - g_{\mu\nu}\overline{\Psi}(\mathrm{i}\gamma^\alpha\overleftrightarrow{\partial}_\alpha - k)\Psi. \tag{102}$$

Its symmetric counterpart is given by

$$\Theta_{\mu\nu} = \mathrm{i}\overline{\Psi}(\gamma_\mu\overleftrightarrow{\partial}_\nu + \gamma_\nu\overleftrightarrow{\partial}_\mu)\Psi - g_{\mu\nu}\overline{\Psi}(\mathrm{i}\gamma^\alpha\overleftrightarrow{\partial}_\alpha - k)\Psi. \tag{103}$$

The component which gives the energy density, or the Hamiltonian density, is given by

$$\Theta_{45} = \mathrm{i}\overline{\Psi}(\gamma^4\overleftrightarrow{\partial}^5 + \gamma^5\overleftrightarrow{\partial}^4)\Psi. \tag{104}$$

5.1. Interaction of a Fermi field with a scalar field

Here we combine the Dirac Lagrangian discussed above with the Klein–Gordon Lagrangian, which turned out to describe the Schrödinger field, with the Fermi and scalar fields interacting through the term $\mathcal{L}_{\text{int.}} = -g\overline{\Psi}\Psi|\Phi|^2$. The full Lagrangian is

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{GDirac}} + \mathcal{L}_{\text{GKG}} + \mathcal{L}_{\text{int.}}\\ &= \overline{\Psi}(\mathrm{i}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu} - k)\Psi - \frac{1}{2m}(\partial^{\mu}\Phi^{*}\partial_{\mu}\Phi - k^2|\Phi|^2) - g\overline{\Psi}\Psi|\Phi|^2.\end{aligned} \tag{105}$$

If we take the variation of this Lagrangian with respect to the field Φ, then we obtain the complex conjugate of the equation

$$\frac{1}{2m}(\partial_{\mu}\partial^{\mu} + k^2)\Phi = g\overline{\Psi}\Psi\Phi \tag{106}$$

which, upon using equations (2) and (4), becomes

$$\mathrm{i}\partial_t\varphi = -\frac{1}{2m}(\nabla^2 + k^2)\varphi + g\overline{\psi}\psi\varphi. \tag{107}$$

The equations of motion follows from Euler–Lagrange equation with respect to $\overline{\Psi}$, and this leads to the Dirac equation with a potential:

$$(\mathrm{i}\gamma^{\mu}\partial_{\mu} - k)\Psi = g\Psi|\Phi|^2. \tag{108}$$

If we use the Dirac matrices of equation (91), with the definition $\chi \equiv g|\Phi|^2 + k$, then the equation above is similar to equation (100) with the constant k replaced by the field χ. With the definitions in equations (2) and (4), this equation becomes similar to the Lévy-Leblond equations (see p 295 of [21]):

$$(\mathrm{i}\sigma\cdot\nabla - \chi)\psi_1 + \sqrt{2}m_{\psi}\psi_2 = 0, \qquad \mathrm{i}\sqrt{2}\partial_t\psi_1 + (\mathrm{i}\sigma\cdot\nabla + \chi)\psi_2 = 0. \tag{109}$$

Note that m_{ψ} denotes the mass of the field Ψ, whereas m is the mass of Φ. If we relate the two components, ψ_1 and ψ_2, by

$$\psi_2 = \frac{1}{\sqrt{2}m\mathrm{i}}(\sigma\cdot\nabla + \mathrm{i}\chi)\psi_1 \tag{110}$$

then the equations above reduce to

$$\left[\mathrm{i}\partial_t + \frac{1}{2m}(\nabla^2 + \chi^2 + \mathrm{i}\sigma\cdot(\nabla\chi))\right]\psi_1 = 0 \tag{111}$$

5.2. Abelian gauge field: Galilean QED

Now we consider the interaction of the Fermi field with the gauge field. Here we consider the Galilean covariant version of the QED Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{GQED}} &= \mathcal{L}_{\text{GDirac}} + \mathcal{L}_{\text{int.}} + \mathcal{L}_{\text{GEM.}}\\ &= \overline{\Psi}(\mathrm{i}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu} - k)\Psi - e\overline{\Psi}\gamma^{\mu}\Psi A_{\mu} - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\\ &= \overline{\Psi}(\mathrm{i}\gamma^{\mu}\overleftrightarrow{D}_{\mu} - k)\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\end{aligned} \tag{112}$$

with the usual definition

$$D_{\mu} \equiv \partial_{\mu} + \mathrm{i}eA_{\mu}. \tag{113}$$

In order to expand this Lagrangian in terms of the embedding in equation (2), we make use of earlier results, namely equations (4), (58), (62) and (94), and we get

$$\begin{aligned}
\mathcal{L}_{\mathrm{GQED}} = {} & \tfrac{1}{2}\big[\big(\nabla\psi_2^\dagger\big)\cdot\sigma\psi_1 - \psi_2^\dagger\sigma\cdot\nabla\psi_1 - \psi_1^\dagger\sigma\cdot\nabla\psi_2 + \big(\nabla\psi_1^\dagger\big)\cdot\sigma\psi_2\big] \\
& - \tfrac{\sqrt{2}}{2}\big(\psi_1^\dagger\partial_t\psi_1 - \big(\partial_t\psi_1^\dagger\big)\psi_1\big) + \mathrm{i}m\sqrt{2}\psi_2^\dagger\psi_2 - \mathrm{i}k\big(\psi_2^\dagger\psi_1 - \psi_1^\dagger\psi_2\big) \\
& - \mathrm{i}e\big(\psi_1^\dagger\sigma\cdot\mathbf{A}\psi_2 + \psi_2^\dagger\sigma\cdot\mathbf{A}\psi_1\big) + \mathrm{i}e\sqrt{2}\big(\phi_m\psi_1^\dagger\psi_1 + \phi_e\psi_2^\dagger\psi_2\big) \\
& - \tfrac{1}{2}(\nabla\times\mathbf{A})^2 + (\nabla\phi_m + \partial_t\mathbf{A})\cdot\nabla\phi_e + \tfrac{1}{2}(\partial_t\phi_e)^2.
\end{aligned} \tag{114}$$

Note that we take $c = 1$, unlike section 4. The Euler–Lagrange equation with respect to $\psi_1^\dagger$ leads to

$$\sigma\cdot(\mathrm{i}\nabla - e\mathbf{A})\psi_2 + \sqrt{2}(\mathrm{i}\partial_t + e\phi_m)\psi_1 + k\psi_2 = 0 \tag{115}$$

and to

$$\sigma\cdot(\mathrm{i}\nabla - e\mathbf{A})\psi_1 - k\psi_1 + \sqrt{2}(m + e\phi_e)\psi_2 = 0 \tag{116}$$

when it is calculated with respect to $\psi_2^\dagger$. This leads to the Lévy-Leblond equations [21] if we take $k = 0$ and choose the magnetic limit, defined in section 4 by taking $\phi_e = 0$.

Now we briefly discuss the opposite procedure, which consists in considering the Euler–Lagrange equations in the (4, 1) manifold, and then defining the embedding. From the variation of $\mathcal{L}_{\mathrm{GQED}}$ with $\overline{\Psi}$, we find

$$(\mathrm{i}\gamma^\mu D_\mu - k)\Psi = (\mathrm{i}\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - k)\Psi = 0. \tag{117}$$

The field equations of motion with respect to A_μ read

$$\partial_\mu F^{\mu\nu} = e\overline{\Psi}\gamma^\nu\Psi. \tag{118}$$

The covariant expansion of these equations is discussed in section 4 of [13]. Therein, it was shown to lead to the Pauli equation, and to describe the correct Landé factor of the electron's intrinsic magnetic moment, as well as the spin–orbit coupling and a term similar to the Darwin term. A thorough investigation of these effects in a Galilean context is found in [22].

6. Concluding remarks

In this paper, we have examined the construction of non-relativistic classical field Lagrangians by enforcing Galilean covariance. The latter is achieved by embedding the Newtonian spacetime into a (4, 1) Minkowski manifold defined by using light-cone coordinates. We have discussed Galilean limits of equations involving Klein–Gordon, Maxwell, Proca and Fermi fields. The formalism provides a straightforward construction of the Lagrangians associated with the electric and magnetic limits of electromagnetism. A similar discussion applies to the spin 1 massive field.

We have previously studied other Galilei-covariant Lagrangian models. We have examined the Bhabha linear wave equation, which describes spin 0 and spin 1 particles in its Duffin–Kemmer–Petiau representation, as well as spin 1/2 particles in its Dirac form, and have shown that the present algorithm leads to known non-relativistic limits of the respective wave equations [12]. Also, various classical fluid models, such as Navier–Stokes equation, Chaplygin gas model, Takahashi model for barotropic irrotational fluid and Thellung–Ziman model for liquid helium have been expressed in a Galilean covariant form in [14].

As explained in the introduction, the five-dimensional formulation of these field theories provides a unified approach to both relativistic and non-relativistic models. Beyond its purely aesthetic appeal, this could lead to practical benefits by deducing Galilean classical solutions from relativistic solutions with one more dimension. A simple example has been considered in section 3. The solution of more intricate equations deserves further investigation. Finally, the similarity between the covariant formulation of Galilei-invariant theories and Poincaré invariant theories means that much of the quantization techniques may be transferred to the non-relativistic regime. Path integral quantization of scalar fields has been considered in [15]. Work on the generalization of section 5.2 to non-Abelian gauge fields, and on the canonical quantization of both scalar and Fermi fields, as well as on the functional quantization of Dirac field, is in progress. Also, a discussion of discrete symmetries, where the reflection $s \to -s$ plays a particular role, will be given elsewhere [16].

Acknowledgments

We are grateful to NSERC (Canada), CNPq (Brazil) and FAPESP (São Paulo, Brazil) for partial financial support. ES acknowledges the hospitality of the Instituto de Física, Universidade Federal da Bahia, where part of this work was done.

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